# Lebesgue Component in Spectrum for Tensor Product of Generic Multiplies

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We present a simple method to construct two so-called generic measure-preserving transformations with Lebesgue component in spectrum of their tensor product. We show how to get two rigid Gaussian systems and two rigid Poisson suspensions with similar spectral properties. "Generic" means here, for instance, that a system $T$ is rigid, rank-one, has all polynomials in its weak close. Generic properties imply simple spectrum of Gaussian suspension $e^T$ ($\exp(T)$). (We call a sums $\sum_{z\in\mathbf{Z}} a_z T^z$ "polynomial" presuming $a_z \geq 0$, and $\sum_z a_z = 1$ in a case of a probability space. For infinite spaces one assumes that $a_z \geq 0$ satisfy the condition $\sum_z a_z \leq 1$.)

To explain our simple idea let's construct for now ergodic transformations $S$, $T$ of an infinite measure space such that

(i) $S$ and $T$ are rigid, (ii) $S \otimes T$ has Lebesgue spectrum,

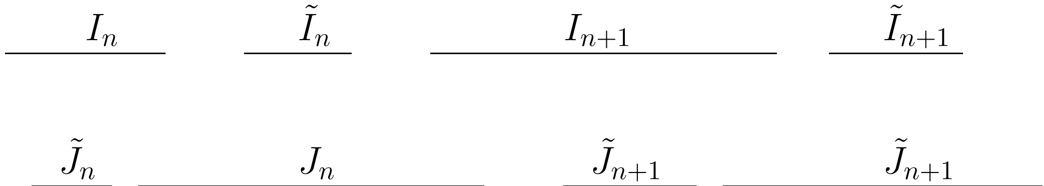
(iii) Poisson suspensions $S_*$, $T_*$ and Gaussian automorphisms $e^S, e^T$ have simple (hence, singular) spectrum.

**Constructions.** We build $T$ and $S$ defining step by step certain integer time-intervals

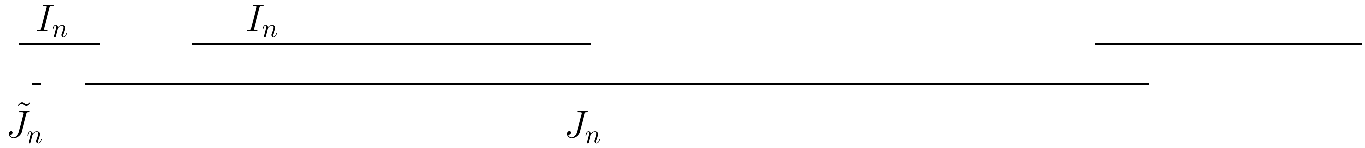
$$\begin{array}{cccccccc} I_1 & \tilde I_1 & I_2 & \tilde I_2 & I_3 & \dots I_n & \tilde I_n & I_{n+1}\dots, \\ \tilde J_1 & J_1 & \tilde J_2 & J_2 & \tilde J_3 & \dots \tilde J_n & J_n & \tilde J_{n+1}\dots, \end{array}$$

satisfying

$$\tilde I_n \subset J_n, \quad \tilde J_n \subset I_n, \qquad \bigcup_n (I_n \cup J_n) = \mathbf{N}.$$




Slightly more realistic picture:



Rank-one transformations $S$,$T$ via large spacers could possess the following property (see arXiv:1106.4655 for rank-one infinite transformations with large spacers). Given functions $f$,$g$ with the supports in fixed towers, respectively, for all large $k$ one has the following zero-correlations:

$$(f, S^n f) = 0,\ n \in I_k,$$

$$(g, T^n g) = 0,\ n \in J_k.$$

As a consequence we get for all large $n$

$$(f \otimes g, S^n f \otimes T^n g) = 0,$$

and, in fact, this is a case for all $f, g$ from a dense family of vectors. Thus, *the product $S \otimes T$ has Lebesgue spectrum.*

We use time intervals $\tilde{I}_n$, $\tilde{J}_n$, $\tilde{I}_n \subset J_n$, $\tilde{J}_n \subset I_n$, to provide the generic behavior of our transformations $S, T$. So, $S$,$T$ could be rigid, of simple spectrum as well as their Gaussian and Poisson suspensions. We see that constructions of *rigid Poisson suspensions $S_*$ and $T_*$, for which $S_* \otimes T_*$ has Lebesgue spectrum in $L_2^0 \otimes L_2^0$,* is reduced to a simple rank-one exercise. In addition, $S_*, T_*$ could be of simple spectrum. We get the same properties for associated Gaussian automorphisms $e^S$, $e^T$. Summarize and formulate

**THEOREM 1.** *There are rigid Poisson suspensions $T_1$,$T_2$ with simple spectrum such that $T_1 \otimes T_2$ has Lebesgue spectrum in the orthocomplement to the sum of the coordinate spaces. There are rigid Gaussian transformations $T_1'$,$T_2'$ with the same spectra as ones of $T_1$,$T_2$.*

**Let $S$ and $T$ act on a probability space.** Below we follow author's draft which has been made public in 2009.

Given vectors $h \in L_2^0$, $\|h\| = 1$, and an automorphism $R$ and $I \subset \mathbf{N}$, denote

$$Corr(R, h, I) := \sum_{n \in I} |(R^n h, h)|.$$

**LEMMA 1.** *Given $\varepsilon > 0$ there are generic maps $S$,$T$, vectors $f, g \in L_2^0(X, \mu)$, $\|f\| = \|g\| = 1$, and integer intervals $I_n, J_n$, $n = 1, 2, \ldots$, such that*

$$\bigcup_n (I_n \cup J_n) = \mathbf{N}.$$

$$Corr(S, f, I_k) < \frac{\varepsilon}{2^{k+1}}, \quad Corr(T, g, J_k) < \frac{\varepsilon}{2^{k+1}},$$

*so*

$$Corr(S \otimes T, f \otimes g, \mathbf{N}) < \varepsilon.$$

**THEOREM 2.** *There are "generic" maps $S$,$T$ such that $S\otimes T$ has Lebesgue component in spectrum.*

Theorem 2 follows obviously from Lemma 1.

Define $\rho(T,\tilde T)=\mu(supp(T^{-1}\tilde T)$.

**LEMMA 2. a)** *Given $T$ with Lebesgue spectrum and $\delta>0$, there exists generic $\tilde T$ such that $\rho(T,\tilde T)<\delta$.*

*b) Given generic $\tilde T$ and $\delta>0$, there exists an automorphism $T$ with Lebesgue spectrum such that $\rho(T,\tilde T)<\delta$.*

Lemma 2 is a consequence of the classic Rokhlin-Kakutani lemma.

**LEMMA 3.** *Let $T$ have Lebesgue spectrum. Given $f\in L_2^0(X,\mu)$, $\|f\|=1$, and $\delta>0$, there exist $M$ and $f'\in L_2^0(X,\mu)$, $\|f'\|=1$, $\|f-f'\|<\delta$ such that*

$$(T^mf',f')=0$$

*for all $m>M$.*

Proof. Consider an orthogonal base $h_{i,j}$, $Th_{i,j}=h_{i+1,j}$, find a finite linear combination $f'$, $\|f-f'\|<\delta$, then set $M=1+max\{\ |i|\ :\ f' \textit{ is not orthogonal to } h_{i,j}\}$.

To build transformations let's apply the following "algorithm". We consider sequences of automorphisms and functions $T_k,\tilde T_k,\ S_k,\tilde S_k,\ f_k,g_k$, $\|f_k\|=\|g'_k\|=1$, such that $S_k,T_k$ have Lebesgue spectrum, $\tilde T_k,\tilde S_k$ are generic,

$$\rho(T_k,\tilde T_k)\to 0,\quad \rho(\tilde T_k,T_{k+1})\to 0,$$

$$\rho(S_k,\tilde S_k)\to 0,\quad \rho(\tilde S_k,S_{k+1})\to 0,$$

$$f_k\to f,\quad g_k\to g,\quad k\to\infty$$

where the rate of the convergences is chosen as fast as we want to provide $T_k\to T$, $S_k\to S$ with

$$Corr(T_k,f_k,J_k)=0,\ Corr(S_k,f_k,I_k)=0,$$
$$Corr(S,f,I_k)<\frac{\varepsilon}{2^{k+1}},\quad Corr(T,g,I_k)<\frac{\varepsilon}{2^{k+1}}.$$

Summarizing the above, we obtain

$$Corr(S\otimes T,f\otimes g,\mathbf{N})<\varepsilon.$$

Thus, $S\otimes T$ has the Lebesgue component in spectrum.

**Generic behavior** of $T$ is inherited by our $\tilde T_k$. On the union $\cup\tilde J_n$ the behavior of powers $T^n$ is generic: for instance, we can get all polynomial limits as $n$ runs within $\cup\tilde J_n$. In addition we provide $T$ to be rank-one. The same thing for $S$. So, $T$ and $S$ will be of simple spectrum, moreover the same property hold for all their symmetric powers. Hence, Gaussian "suspensions" $e^S$ and $e^T$ are rigid, have simple spectrum. Their spectra are "very" singular, but the product $e^S\otimes e^T=$

$e^{S\oplus T}$ has Lebesgue component in spectrum (we do not prove that in fact it is Lebesgue in $L_2^0 \otimes L_2^0$).

**REMARK.** A similar effect via different methods is presented by V. Bergelson, A. del Junco, M. Lemanczyk, J. Rosenblatt in arXiv:1103.0905. There the authors assert (Proposition 3.35) **for arbitrary** rigid, weakly mixing dynamical system $T$ the existence of a such $S$ that in the orthocomplement of the sum of multiplier spaces the product $T \otimes S$ has absolutely continuous spectrum.